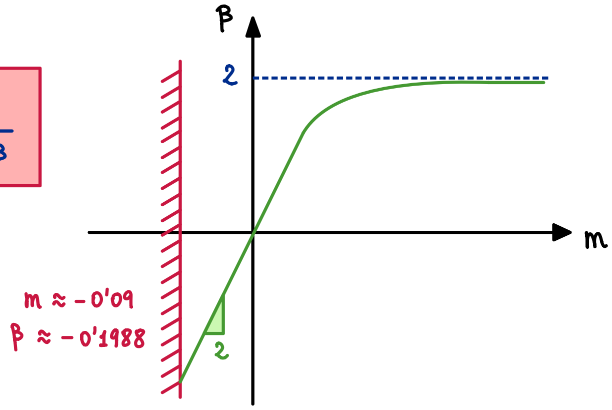
Capa Límite de Falkner-Skan

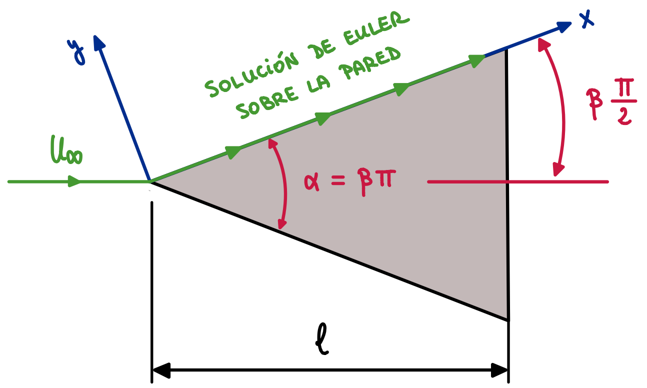
Podemos encontrar soluciones autosemejantes de las ecuaciones de CL con $\frac{du_e}{dx} \neq 0$ (CON GRADIENTE DE PRESIONES), y son relevantes.

$$\boxed{u_e = A x^m}$$

$$\boxed{\beta = \frac{2m}{1+m} \rightleftarrows m = \frac{\beta}{2-\beta}}$$

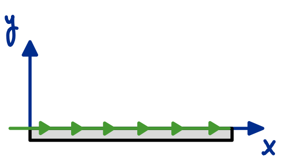
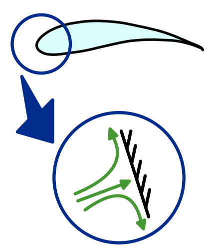
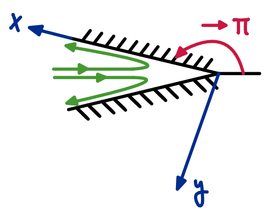
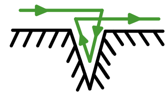
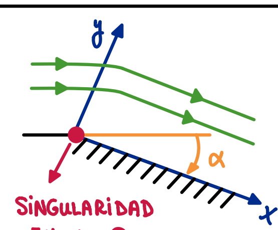


Obtención de A (suponiendo $0 \le m \le 1$):

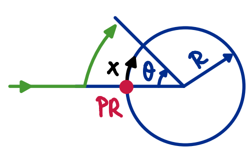


$$\frac{u_e}{U_\infty} = g(\beta)\left(\frac{x}{\ell}\right)^m \longrightarrow u_e = \underbrace{U_\infty \frac{g(\beta)}{\ell^m}}_{A} x^m \longrightarrow$$

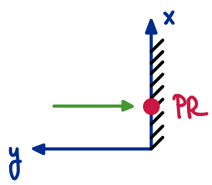
$$\longrightarrow \boxed{A = U_\infty \frac{g(\beta)}{\ell^m}}$$

	I	II	III	IV
β	0	1	$\to 2$	$-0'1988 < \beta < 0$
m	0	1	$\to \infty$	$-0'09 < m < 0$
α	0	$\frac{\pi}{2}$	$\to \pi$	$-17^\circ < \alpha < 0^\circ$
ESQUEMA	 PLACA PLANA $u_e = A = cte$ (BLASIUS)	 FLUJO DE PUNTO DE REMANSO $u_e = Ax$	 FLUJO EN CAVIDAD ROTACIONAL 	 SINGULARIDAD EN $x = 0$ FLUJO SOBRE ESQUINA EN EXPANSIÓN m muy negativo: separación $m > -0'09$

Para el caso de un cilindro:



$$u_e = 2U_\infty \operatorname{sen}\theta \xrightarrow{\theta << 1} u_e \approx 2U_\infty\theta \overset{\theta = \frac{x}{R}}{=} \underbrace{\frac{2U_\infty}{R}}_{A} x^1 \longrightarrow$$



$$\longrightarrow \boxed{A_{CILINDRO} = \frac{2U_\infty}{R}} \quad \boxed{m_{CILINDRO} = 1}$$

NOTA

$$U_e \frac{dU_e}{dx} = Ax^m \cdot Amx^{m-1} = mA^2x^{2m-1} = \underbrace{\left(Ax^m\right)^2}_{U_e^2} m x^{-1} \longrightarrow \boxed{U_e \frac{dU_e}{dx} = \frac{mU_e^2}{x} = \frac{\beta}{2-\beta}\frac{U_e^2}{x}}$$

El problema a resolver es:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\beta}{2-\beta}\frac{U_e^2}{x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$y = 0: \; u = 0 \; ; \; v = 0$

$y \to \infty: \; u \to U_e(x) = Ax^m$

$x = 0: \; u = U_e\big|_{x=0}$

EXISTE SOLUCIÓN DE SEMEJANZA

Introducimos la función de corriente ψ:

$$\boxed{\psi = \sqrt{(2-\beta)\,\nu\, x\, U_e(x)}\; f(\eta)} \qquad \boxed{\eta = y\sqrt{\frac{U_e(x)}{(2-\beta)\,\nu\, x}}}$$

$(2-\beta) \longrightarrow$ Simplificará la EDO que obtendremos

Derivadas de la variable de semejanza respecto a x, y:

$$\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial x}\left\{ y\sqrt{\frac{U_e(x)}{(2-\beta)\,\nu\, x}}\right\} = \frac{\partial}{\partial x}\left\{ y\sqrt{\frac{Ax^{m-1}}{(2-\beta)\,\nu\, x}}\right\} = \sqrt{\frac{A}{(2-\beta)\,\nu}}\; y\, \frac{m-1}{2}\, x^{\frac{m-1}{2}-1} =$$

$$= \frac{m-1}{2}\underbrace{y\sqrt{\frac{Ax^m}{(2-\beta)\,\nu\, x}}}_{\eta}\;\underbrace{x^{\frac{m-1}{2}-1-\frac{m-1}{2}}}_{x^{-1}} = \frac{m-1}{2}\frac{\eta}{x} \longrightarrow \boxed{\frac{\partial \eta}{\partial x} = \frac{\beta-1}{2-\beta}\frac{\eta}{x}}$$

$$\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial y}\left\{ y\sqrt{\frac{U_e(x)}{(2-\beta)\,\nu\, x}}\right\} = \sqrt{\frac{U_e(x)}{(2-\beta)\,\nu\, x}} = \frac{\eta}{y} \longrightarrow \boxed{\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_e(x)}{(2-\beta)\,\nu\, x}} = \frac{\eta}{y}}$$

Como la ecuación de continuidad se satisface automáticamente por trabajar con la función de corriente, nos centramos en la ECdM_x:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\beta}{2-\beta}\frac{U_e^2}{x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$\boxed{u}$

$$u = \frac{\partial\psi}{\partial y} = \frac{\partial}{\partial y}\left[\sqrt{(2-\beta)\nu x U_e(x)}\; f(\eta)\right] = \sqrt{(2-\beta)\nu x U_e(x)}\;\underbrace{\frac{df}{d\eta}}_{f'}\frac{\partial\eta}{\partial y} =$$

$$= \sqrt{(2-\beta)\nu x U_e(x)}\; f' \sqrt{\frac{U_e(x)}{(2-\beta)\nu x}} = U_e(x)\, f' \longrightarrow \boxed{u = U_e(x)\, f'}$$

$\boxed{v}$

$$v = -\frac{\partial\psi}{\partial x} = -\frac{\partial}{\partial x}\left[\sqrt{(2-\beta)\nu x U_e(x)}\; f(\eta)\right] = -\sqrt{(2-\beta)\nu}\left[\frac{U_e(x) + \overbrace{\boxed{x\frac{dU_e(x)}{dx}}}^{\frac{\beta}{2-\beta}U_e(x)}}{2\sqrt{x U_e(x)}}\, f + \right.$$

$$\left. + \sqrt{x U_e(x)}\;\underbrace{\frac{df}{d\eta}}_{f'}\frac{\partial\eta}{\partial y}\right] = -\sqrt{(2-\beta)\nu}\left[\sqrt{\frac{U_e(x)}{x}}\, f + \frac{\beta-1}{2-\beta}\frac{\eta}{x}\, f' \sqrt{\frac{U_e(x)}{x}}\right] =$$

$$= \sqrt{\frac{\nu U_e(x)}{(2-\beta)x}}\left[-f + (1-\beta)\eta f'\right] \longrightarrow \boxed{v = \sqrt{\frac{\nu U_e(x)}{(2-\beta)x}}\left[(1-\beta)\eta f' - f\right]}$$

$\boxed{\frac{\partial u}{\partial x}}$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[U_e(x)\, f'\right] = \overbrace{\frac{dU_e(x)}{dx}}^{\frac{\beta}{2-\beta}\frac{U_e(x)}{x}}\, f' + U_e(x)\, f' \frac{\partial\eta}{\partial x} = \frac{\beta}{2-\beta}\frac{U_e(x)}{x}\, f' + U_e(x)\, f' \frac{\beta-1}{2-\beta}\frac{\eta}{x} =$$

$$= \frac{1}{2-\beta}\frac{U_e(x)}{x}\left[\beta f' - (1-\beta)\eta f''\right] \longrightarrow \boxed{\frac{\partial u}{\partial x} = \frac{1}{2-\beta}\frac{U_e(x)}{x}\left[\beta f' - (1-\beta)\eta f''\right]}$$

$\boxed{\frac{\partial u}{\partial y}}$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[U_e(x)\, f'\right] = U_e(x)\, f'' \frac{\partial\eta}{\partial y} = U_e(x)\, f'' \sqrt{\frac{U_e(x)}{(2-\beta)\nu x}} \longrightarrow \boxed{\frac{\partial u}{\partial y} = U_e(x)\sqrt{\frac{U_e(x)}{(2-\beta)\nu x}}\, f''}$$

$$\boxed{\frac{\partial^2 u}{\partial y^2}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left[U_e(x)\, f''\sqrt{\frac{U_e(x)}{(2-\beta)\nu x}}\right] = U_e(x)\sqrt{\frac{U_e(x)}{(2-\beta)\nu x}}\, f'''\frac{\partial \eta}{\partial y} =$$

$$= U_e(x)\sqrt{\frac{U_e(x)}{(2-\beta)\nu x}}\, f'''\sqrt{\frac{U_e(x)}{(2-\beta)\nu x}} = \frac{U_e^2(x)}{(2-\beta)\nu x} f''' \longrightarrow \boxed{\frac{\partial^2 u}{\partial y^2} = \frac{U_e^2(x)}{(2-\beta)\nu x} f'''}$$

Sustituyendo en la $ECdM_x$:

$$\frac{U_e(x)\, f'}{2-\beta}\frac{U_e(x)}{x}\left[\beta f' - (1-\beta)\eta f''\right] + \sqrt{\frac{\nu\, U_e(x)}{(2-\beta)x}}\left[(1-\beta)\eta f' - f\right] U_e(x)\sqrt{\frac{U_e(x)}{(2-\beta)\nu x}} f'' =$$

$$= \frac{\beta}{2-\beta}\frac{U_e^2(x)}{x} + \nu\frac{U_e^2(x)}{(2-\beta)\nu x} f'''$$

$$\frac{\beta}{2-\beta}\frac{U_e^2(x)}{x}(f')^2 - \cancel{\frac{1-\beta}{2-\beta}\frac{U_e^2(x)}{x}\eta f' f''} + \cancel{\frac{1-\beta}{2-\beta}\frac{U_e^2(x)}{x}\eta f' f''} - \frac{1}{2-\beta}\frac{U_e^2(x)}{x} f f'' = \frac{\beta}{2-\beta}\frac{U_e^2(x)}{x} + \frac{1}{2-\beta}\frac{U_e^2(x)}{x} f'''$$

$$\frac{1}{2-\beta}\frac{U_e^2(x)}{x} f''' + \frac{1}{2-\beta}\frac{U_e^2(x)}{x} f f'' - \frac{\beta}{2-\beta}\frac{U_e^2(x)}{x}(f')^2 + \frac{\beta}{2-\beta}\frac{U_e^2(x)}{x} = 0$$

$$\boxed{f''' + f f'' + \beta\left[1 - (f')^2\right] = 0}$$

→ Diferencia con Blasius

Respecto a las condiciones de contorno:

$y = 0: \; u = 0 \; ; \; v = 0$

$y \to \infty: \; u \to U_e(x) = A x^m$

$x = 0: \; u = U_e\big|_{x=0}$

$\Longrightarrow$

$\eta = 0: \; f = 0 \; ; \; f' = 0$

$\eta \to \infty: \; f' \to 1$

$\eta \to \infty: \; f' \to 1$

} COLAPSAN

El problema queda:

$$\boxed{\begin{array}{c} f''' + f f'' + \beta\left[1-(f')^2\right] = 0 \\ \eta = 0: \ f = 0;\ f' = 0 \\ \eta \to \infty: \ f' \to 1 \end{array}}$$

$$\Big\downarrow \quad \text{VALOR DE CONTORNO} \longrightarrow \text{VALOR INICIAL}$$

$$\boxed{\begin{array}{c} f''' + f f'' + \beta\left[1-(f')^2\right] = 0 \\ \eta = 0: \ f = 0;\ f' = 0 \\ f'' = f_0''(\beta) \Big/ \lim_{\eta\to\infty} f' = 1 \end{array}}$$

Vamos a obtener C_f, δ_1, δ_2 y H_{12}.

Coeficiente de fricción:

$$\tau_p = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu\, U_e(x)\sqrt{\frac{U_e(x)}{(2-\beta)\,\nu\, x}}\, f_0'' = \rho\, U_e^2(x)\sqrt{\frac{\nu}{(2-\beta)\,U_e(x)\, x}}\, f_0''$$

$$C_f(x) = \frac{\tau_p(x)}{\frac{1}{2}\rho U_e^2(x)} = 2\sqrt{\frac{\nu}{(2-\beta)\,U_e(x)\,x}}\, f_0'' = \frac{2 f_0''}{\sqrt{2-\beta}}\, Re_x^{-1/2} \longrightarrow$$

$$\longrightarrow \boxed{C_f(x) = \frac{2 f_0''}{\sqrt{2-\beta}}\, Re_x^{-1/2} = \sqrt{2(1+m)}\; f_0''\, Re_x^{-1/2}}$$

Espesor de desplazamiento :

$$dy\Big|_{x=cte} = \sqrt{\frac{(2-\beta)\,\nu x}{U_\infty}}\,d\eta$$

$$\delta_1 = \int_0^\infty \Big(1 - \underbrace{\frac{u}{U_e(x)}}_{f'}\Big)\,dy = \sqrt{\frac{(2-\beta)\,\nu x}{U_\infty}} \int_0^\infty \left(1 - f'\right) d\eta = \sqrt{\frac{(2-\beta)\,\nu x}{U_\infty}}\,\left(\eta - f\right)_{\eta\to\infty} \longrightarrow$$

$$\longrightarrow \boxed{\frac{\delta_1}{x} = \sqrt{2-\beta}\,\left(\eta - f\right)_{\eta\to\infty} Re_x^{-1/2}}$$

Espesor de cantidad de movimiento :

$$\delta_2 = \int_0^\infty \frac{u}{U_e(x)}\left(1 - \frac{u}{U_e(x)}\right) dy = \sqrt{\frac{(2-\beta)\,\nu x}{U_\infty}} \int_0^\infty f'\left(1 - f'\right) d\eta = \sqrt{\frac{(2-\beta)\,\nu x}{U_\infty}} \left[\, f\Big|_0^\infty - \underbrace{\int_0^\infty \left(f'\right)^2 d\eta}_{?}\,\right]$$

$$f''' + ff'' + \beta\left[1 - (f')^2\right] = 0$$

$$\left\{ \begin{array}{ll} u = f' \longrightarrow & du = f''\,d\eta \\ dv = f'\,d\eta \longrightarrow & v = f \end{array} \right\}$$

$$ff'' = -\beta\left[1 - (f')^2\right] - f'''$$

$$\int_0^\infty \left(f'\right)^2 d\eta = ff'\Big|_0^\infty - \int_0^\infty ff''\,d\eta = f\Big|_{\eta\to\infty} + \beta\int_0^\infty d\eta - \beta\int_0^\infty \left(f'\right)^2 d\eta + \int_0^\infty f'''\,d\eta$$

$$(1+\beta)\int_0^\infty \left(f'\right)^2 d\eta = \underbrace{f\Big|_{\eta\to\infty} - \int_0^\infty d\eta}_{-(\eta - f)_{\eta\to\infty}} + (1+\beta)\int_0^\infty d\eta + \underbrace{\int_0^\infty f'''\,d\eta}_{-f''_0(\beta)}$$

$$\boxed{\int_0^\infty \left(f'\right)^2 d\eta = -\frac{(\eta - f)_{\eta\to\infty} + f''_0(\beta)}{1+\beta} + \int_0^\infty d\eta}$$

Entonces :

$$\delta_2 = \sqrt{\frac{(2-\beta)\nu x}{U_\infty}} \left[f\Big|_0^\infty - \int_0^\infty (f')^2 \, d\eta \right]$$

$$\delta_2 = \sqrt{\frac{(2-\beta)\nu x}{U_\infty}} \left[\underbrace{f\Big|_0^\infty - \int_0^\infty d\eta}_{-(\eta - f)_{\eta\to\infty}} + \frac{(\eta - f)_{\eta\to\infty} + f_0''(\beta)}{1+\beta} \right]$$

$$\delta_2 = \sqrt{\frac{(2-\beta)\nu x}{U_\infty}} \left[(\eta - f)_{\eta\to\infty} + \frac{(\eta - f)_{\eta\to\infty} + f_0''(\beta)}{1+\beta} \right]$$

$$\delta_2 = \sqrt{\frac{(2-\beta)\nu x}{U_\infty}} \left[\frac{f_0''(\beta)}{1+\beta} + (\eta - f)_{\eta\to\infty} \left(\frac{1}{1+\beta} - 1 \right) \right]$$

$$\delta_2 = \sqrt{\frac{(2-\beta)\nu x}{U_\infty}} \left[\frac{f_0''(\beta)}{1+\beta} - \frac{\beta}{1+\beta} (\eta - f)_{\eta\to\infty} \right]$$

$$\boxed{\frac{\delta_2}{x} = \frac{\sqrt{2-\beta}}{1+\beta} \left[f_0''(\beta) - \beta(\eta - f)_{\eta\to\infty} \right] Re_x^{-1/2}}$$

Factor de forma :

$$H_{12} = \frac{\delta_1}{\delta_2} = \frac{\frac{\delta_1}{x}}{\frac{\delta_2}{x}} = \frac{\sqrt{2-\beta}\,(\eta - f)_{\eta\to\infty} Re_x^{-1/2}}{\frac{\sqrt{2-\beta}}{1+\beta} \left[f_0''(\beta) - \beta(\eta - f)_{\eta\to\infty} \right] Re_x^{-1/2}}$$

$$\boxed{H_{12} = \frac{(1+\beta)(\eta - f)_{\eta\to\infty}}{f_0''(\beta) - \beta(\eta - f)_{\eta\to\infty}}}$$

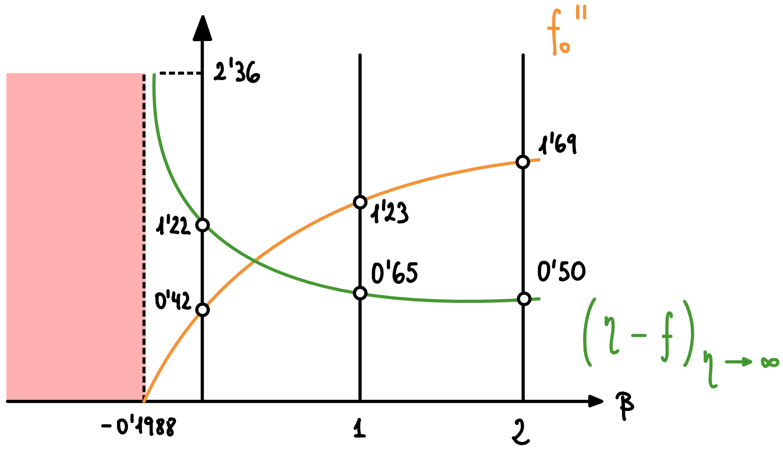
Hay otro parámetro interesante : $\dfrac{\delta_2^2 \dfrac{dU_e}{dx}}{\nu}$

$$\frac{\delta_2^2 \frac{dU_e}{dx}}{\nu} = \frac{x^2}{\nu} \frac{2-\beta}{(1+\beta)^2} \left[f_0''(\beta) - \beta(\eta - f)_{\eta\to\infty} \right]^2 \frac{\nu}{U_e(x)\, x} \frac{\beta}{2-\beta} \frac{U_e(x)}{x}$$

$$\boxed{\frac{\delta_2^2 \frac{dU_e}{dx}}{\nu} = \frac{\beta}{(1+\beta)^2} \left[f_0''(\beta) - \beta(\eta - f)_{\eta\to\infty} \right]^2}$$

$$\boxed{\frac{\delta_2^2 \frac{dU_e}{dx}}{\nu} \sim \frac{\delta_2^2 \frac{U_e}{x}}{\nu} \sim \underbrace{\left(\frac{\delta_2}{x}\right)^2}_{\sim Re^{-1}} \underbrace{\frac{U_e\, x}{\nu}}_{\sim Re} \sim cte \quad (\text{para cada } \beta)}$$

Resumen :



α	β	m	f_0''	$(\eta - f)_{\eta\to\infty}$	$C_f\, Re_x^{1/2}$	$\frac{\delta_1}{x} Re_x^{1/2}$	$\frac{\delta_2}{x} Re_x^{1/2}$	H_{12}	$\frac{\delta_2^2}{\nu}\frac{dU_e}{dx}$
≈ 17°	- 0'1988	- 0'09	0	2'36	0	3'51	0'87	4'03	- 0'068
0°	0	0	0'467	1'22	0'66	1'72	0'66	2'59	0
90°	1	1	1'23	0'65	2'46	0'65	0'29	2'21	0'085
→ 180°	→ 2	→ 1'69	→ 0'5	→ 0'5	→ 0	→ 0	→ 0	→ 2'16	→ 0'106

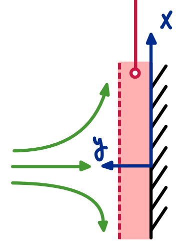
$f_0'' = 0 \rightleftarrows C_f = 0$
(SEPARACIÓN)

$\uparrow C_f$ porque se va llenando el perfil de velocidades

Capa límite cerca de un punto de remanso $\left(\beta = m = 1\ ;\ U_e = A x\ ;\ \frac{dU_e}{dx} = A\right)$:

$$\frac{\delta_2^2 \, \frac{dU_e}{dx}}{\nu} = \frac{\delta_2^2 \, A}{\nu} \sim cte \longrightarrow \begin{cases} \delta_2^2 = 0'85 \, \frac{\nu}{A} = cte \\ \delta_1^2 = H_{12}^2 \, \delta_2^2 = cte \end{cases}$$

En un ala cerca del BA no hay CL, sino una región de efectos viscosos con δ_1 y δ_2 ctes



Perfil supercrítico de un A320 en aproximación:

$\frac{R_c}{c} \sim 0'10$

$c \sim 4\ m$

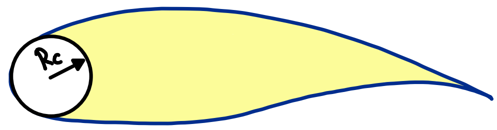
$U_\infty \sim 80\ m/s$

$\nu \sim 1'5 \cdot 10^{-5}\ m^2/s$

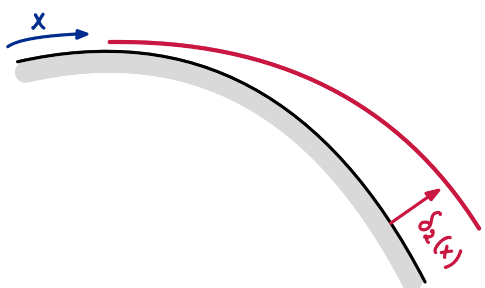
$$U_e = 2U_\infty \frac{x}{R_c}$$

$$A = \frac{2U_\infty}{R_c} = \frac{20\,U_\infty}{c}$$

$$\left(\frac{\delta_2}{c}\right)^2 \approx \frac{0'085}{20}\,\frac{\nu}{U_\infty c} \longrightarrow \frac{\delta_2}{c} \sim \frac{1}{2\cdot 10^4} \longrightarrow \boxed{\delta_2 \sim 0'2\ mm}$$



NOTA



En el caso de $\frac{dU_e}{dx} < 0$ (difusión), si el gradiente adverso se intensifica, entonces $\frac{dU_e}{dx}$ se irá haciendo más negativo $\xrightarrow[\frac{\delta_2^2 \frac{dU_e}{dx}}{\nu} \sim cte]{}$ δ_2 crece $\overset{\cdots}{\longrightarrow}$ SEPARACIÓN

Suponiendo la placa plana POROSA, la solución de C.L. sigue siendo autosemejante si la velocidad de succión/soplado es de la forma:

$$\frac{v_p}{U_e(x)} = \sqrt{\frac{\nu}{(2-\beta)\,U_e(x)\,X}}\left[(1-\beta)\,\eta f' - f\right] \xrightarrow[\text{semejanza y el escalado}]{\text{Respetando la solución de}} \begin{cases} \frac{v_p}{U_e(x)} = \sqrt{\frac{\nu}{(2-\beta)\,U_e(x)\,X}}\ \tilde{v}_p \\ \tilde{v}_p = O(1) \end{cases}$$

Y el problema sería:

$$\boxed{\begin{gathered} f''' + f f'' + \beta\left[1 - (f')^2\right] = 0 \\ \eta = 0:\ f = -\tilde{v}_p\ ;\ f' = 0 \\ f'' = f_0''(\beta, \tilde{v}_p) \Big/ \lim_{\eta\to\infty} f' = 1 \end{gathered}} \longrightarrow f = f(\eta\,;\beta, \tilde{v}_p)$$